

# Critical mass renormalization in renormalized $\phi^4$ theories in two and three dimensions

Andrea Pelissetto<sup>1</sup> and Ettore Vicari<sup>2</sup>

<sup>1</sup> *Dipartimento di Fisica dell'Università di Roma "La Sapienza" and INFN, Sezione di Roma I, P.le Aldo Moro 2, I-00185 Roma, Italy and*

<sup>2</sup> *Dipartimento di Fisica dell'Università di Pisa and INFN, Largo Pontecorvo 3, I-56127 Pisa, Italy*  
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We consider the  $O(N)$ -symmetric  $\phi^4$  theory in two and three dimensions and determine the nonperturbative mass renormalization needed to obtain the  $\phi^4$  continuum theory. The required nonperturbative information is obtained by resumming high-order perturbative series in the massive renormalization scheme, taking into account their Borel summability and the known large-order behavior of the coefficients. The results are in good agreement with those obtained in lattice calculations.

## I. INTRODUCTION

The  $N$ -vector  $\phi^4$  theory for a field  $\phi(x)$  with  $N$  components, with Lagrangean

$$\mathcal{L} = \frac{1}{2} \sum_{\mu} \partial_{\mu} \phi^2 + \frac{1}{2} \mu_0^2 \phi^2 + \frac{1}{4!} g \phi^4 \quad (1)$$

(where  $\phi^2 \equiv \phi \cdot \phi$  and  $\phi^4 \equiv (\phi^2)^2$ ), is an important field theory model that can be used to describe a wide variety of systems under critical conditions. Because of the ultraviolet divergences, a proper definition requires the introduction of a regularization. We will use here the corresponding lattice theory that has the advantage of being well defined also at the nonperturbative level. On a regular  $d$ -dimensional cubic lattice the action is given by

$$S = \frac{1}{2} \sum_{xy} J(x-y) \phi_x \phi_y + \sum_x \left( \frac{1}{2} \mu_0^2 \phi_x^2 + \frac{1}{4!} g \phi_x^4 \right), \quad (2)$$

where the fields  $\phi_x$  are  $N$ -dimensional vectors. We assume that fields and  $\mu_0^2$  are chosen so that the Fourier transform  $\tilde{J}(p)$  satisfies  $\tilde{J}(p) = p^2 + O(p^4)$  for small momenta. We make no other assumption on  $J(x)$ , so that model (2) represents the most general lattice model consistent with the  $\phi^4$  continuum theory.

In this paper we shall focus on the theory in  $d = 2$  and  $d = 3$ . In this case the model is superrenormalizable, which greatly simplifies the determination of the continuum limit. Indeed, it is enough to perform a (nonperturbative) mass renormalization. If one defines a renormalized mass  $t = \mu_0^2 - \mu_{0c}^2$ , the continuum limit is obtained by considering  $g \rightarrow 0$ ,  $t \rightarrow 0$  at fixed  $tg^{2/(4-d)}$ , where  $d$  is the space dimension. In the statistical-mechanics framework,  $\mu_{0c}^2$  represents the value of the bare parameter  $\mu_0^2$  at which the statistical system undergoes a continuous second-order transition. The determination of  $\mu_{0c}^2$  is crucial, as it represents a prerequisite in any study of the  $\phi^4$  theory in the continuum limit. Beside its field-theoretical interest,  $\mu_{0c}^2$  is also required in some calculations concerning dilute relativistic and nonrelativistic Bose gases, in homogeneous conditions and in the presence of trapping potentials [1–4].

The determination of  $\mu_{0c}^2$  in the limit  $g \rightarrow 0$  is not an easy task as it represents a nonperturbative renormalization. In two dimensions it has been computed either by Monte Carlo simulations of lattice models or by an analysis of the corresponding Hamiltonian model defined in one dimension [5–11]. In three dimensions results, obtained by means of Monte Carlo simulations, are available for  $N = 2$  [2]. Here, we will perform a different calculation, using the high-order perturbative series, computed in the massive renormalization scheme, that provide the critical exponents for the critical theory [12, 13]. The resummation of these perturbative series [14], taking into account their Borel summability and the known large-order behavior of the coefficients [15] allows us to obtain the nonperturbative information that is needed to compute  $\mu_{0c}^2$ . As we shall see, we obtain results for  $d = 2$  and  $d = 3$  with a precision that is comparable with that obtained using state-of-the-art numerical simulations of lattice models, confirming the accuracy of resummed perturbation theory.

## II. TWO DIMENSIONS

Let us consider the generic  $\phi^4$  model (2) on a two-dimensional square lattice. We only discuss the case  $N = 1$ , as only for this value of  $N$  the model undergoes a standard transition from a symmetric to a broken phase. The superrenormalizability properties of the theory allow us to predict

$$\mu_{0c}^2 = Ag \ln g + Bg + O(g^2 \ln^2 g), \quad (3)$$

for  $g \rightarrow 0$ . The constant  $A$  can be computed in perturbation theory and does not depend on the regularization, while the constant  $B$  is nonperturbative and regularization dependent.

To determine  $A$  and  $B$  we consider the integrated bare two-point correlation function

$$\chi = \sum_x \langle \phi_0 \phi_x \rangle. \quad (4)$$

In [16–18] it was shown that the combination  $\tilde{\chi} = \chi g$  is a regularization-independent function  $F_{\chi}(t)$  of  $t = t/g$ ,

$t = \mu_0^2 - \mu_{0c}^2$  in the limit  $g \rightarrow 0$ ,  $t \rightarrow 0$  at fixed  $\tilde{t}$ . This limit, which was called critical crossover limit [18–20], corresponds to what we call continuum limit in the present context. The quantity  $\tilde{t}$  is the dimensionless renormalized mass: for  $\tilde{t} \rightarrow 0$  we obtain the critical massless regime, while for  $\tilde{t} \rightarrow \infty$  we recover the weak-coupling behavior.

The function  $F_\chi(\tilde{t})$  is intrinsically nonperturbative. In [18] it was determined by resumming the perturbative series of renormalization-group invariant functions in the massive renormalization scheme. Four-loop results were used [12], taking explicitly into account [14] the Borel summability of the perturbative series and the large-order behavior of their coefficients, determined by non-perturbative instanton calculations [15]. For  $\tilde{t} \rightarrow \infty$  reference [18] obtained

$$F_\chi(\tilde{t}) = \frac{1}{\tilde{t}} + \frac{1}{\tilde{t}^2} \left[ \frac{1}{8\pi} \ln \left( \frac{8\pi\tilde{t}}{3} \right) + \frac{3}{8\pi} + D_2 \right] + O(\tilde{t}^{-3} \ln^2 \tilde{t}), \quad (5)$$

where  $D_2$  is a nonperturbative constant. Resummation of the perturbative series gave  $D_2 = -0.0524(2)$ .

Let us now compute  $\chi$  in the lattice model. At one loop we have

$$\chi = \frac{1}{\mu_0^2} - \frac{g}{2\mu_0^4} \int \frac{d^2 p}{(2\pi)^2} \frac{1}{\tilde{J}(p) + \mu_0^2} + O(g^2). \quad (6)$$

We now rewrite  $\mu_0^2 = t + \mu_{0c}^2$  and expand all quantities for  $g \rightarrow 0$ . Since

$$\int \frac{d^2 p}{(2\pi)^2} \frac{1}{\tilde{J}(p) + \mu_0^2} = -\frac{1}{4\pi} \ln \mu_0^2 + K + O(\mu_0^2 \ln \mu_0^2), \quad (7)$$

where  $K$  is a constant that depends on  $J(x)$ , we obtain

$$\tilde{\chi} \approx \frac{1}{\tilde{t}} - \frac{\ln g}{\tilde{t}^2} \left( A - \frac{1}{8\pi} \right) + \frac{1}{8\pi\tilde{t}^2} \ln \tilde{t} + \frac{1}{2\tilde{t}^2} (K - 2B), \quad (8)$$

where we have replaced  $t$  with  $\tilde{t} = t/g$ . Comparison with (5) gives

$$\begin{aligned} A &= \frac{1}{8\pi}, \\ B &= -\frac{3}{8\pi} - \frac{1}{8\pi} \ln \frac{8\pi}{3} - \frac{K}{2} - D_2. \end{aligned} \quad (9)$$

The constant  $B$  depends on the regularization through the constant  $K$ , as expected for a bare mass term.

To compare with the results reviewed in [11], let us introduce the perturbatively renormalized mass  $\mu^2$  defined by

$$\mu_0^2 = \mu^2 - \frac{g}{2} \int \frac{d^2 p}{(2\pi)^2} \frac{1}{\tilde{J}(p) + \mu^2}. \quad (10)$$

We wish now to compute the critical value  $\mu_c^2$  that corresponds to  $\mu_{0c}^2$ . We find  $\mu_c^2 = Cg$ , where  $C$  is independent

of the regularization. Using (9) and (7) we find that  $C$  satisfies the equation

$$C + \frac{1}{8\pi} \ln C = -D_2 - \frac{3}{8\pi} - \frac{1}{8\pi} \ln \frac{8\pi}{3}. \quad (11)$$

Note that  $K$  cancels out, proving the regularization independence of  $C$ . Solving this equation we obtain

$$C = 0.01515(6), \quad (12)$$

where the reported error is related to the uncertainty on  $D_2$ . Taking into account the different normalization of the coupling constant, we obtain for the quantity  $f_0$  defined in [11]

$$f_0 = \frac{1}{6C} = 11.00(4). \quad (13)$$

The quantity  $f_0$  has been also computed by means of other techniques [5–11]. References [7, 8, 11] use Monte Carlo methods, while [9, 10] consider the Hamiltonian quantum formulation in one dimension. Results are reviewed in [11]. The most recent estimates ( $f_0 = 10.92(13), 11.06(2), 11.88(56), 11.15(9)$  of [8], [9], [10], and [11], respectively) are all in good agreement with our result. Note also that the error on the estimate (13) is comparable with those obtained using state-of-the-art numerical algorithms, confirming the accuracy of resummed perturbation theory.

### III. THREE DIMENSIONS

Analogous considerations apply to three dimensions. Using the results of [18] we are now going to compute the nonperturbative mass renormalization for the three-dimensional theory. We start from the two-loop expansion of  $\chi$  in powers of the bare coupling constant  $g$ :

$$\begin{aligned} \chi^{-1} &= \mu_0^2 + \frac{N+2}{6} g T_1(\mu_0^2) - \frac{N+2}{18} g^2 T_3(\mu_0^2) \\ &\quad - \left( \frac{N+2}{6} \right)^2 g^2 T_1(\mu_0^2) T_2(\mu_0^2), \end{aligned} \quad (14)$$

where

$$\begin{aligned} T_1(m^2) &= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\Delta(p)}, \\ T_2(m^2) &= \int \frac{d^3 p}{(2\pi)^3} \frac{1}{\Delta(p)^2}, \\ T_3(m^2) &= \int \frac{d^3 p}{(2\pi)^3} \frac{d^3 q}{(2\pi)^3} \frac{1}{\Delta(p)\Delta(q)\Delta(p+q)}. \end{aligned} \quad (15)$$

with  $\Delta(p) = \tilde{J}(p) + m^2$ . The continuum limit is obtained by tuning  $\mu_0^2$  to the critical value  $\mu_{0c}^2$ . More precisely,  $\tilde{\chi} = \chi g^2$  becomes a universal function of the renormalized mass  $\tilde{t} = t/g^2$ , where  $t = \mu_0^2 - \mu_{0c}^2$ , for  $t \rightarrow 0$ ,  $g \rightarrow 0$  at fixed  $\tilde{t}$ .

Let us now proceed as in two dimensions. We expand

$$\mu_{0c}^2 = Ag + Bg^2 \ln g + Cg^2, \quad (16)$$

rewrite  $\mu_0^2 = t + \mu_{0c}^2$  and expand in powers of  $g$ . Now, for  $m \rightarrow 0$  we have

$$\begin{aligned} T_1(m^2) &= T_1(0) - \frac{m}{4\pi} + K_1 m^2 + O(m^3) \\ T_2(m^2) &= \frac{1}{8\pi m} - K_1 + O(m) \\ T_3(m^2) &= -\frac{1}{32\pi^2} \ln m^2 + K_2 + O(m). \end{aligned} \quad (17)$$

Using these expressions, for  $t, g \rightarrow 0$  we obtain the expansion

$$\begin{aligned} \tilde{\chi}^{-1} &= g^{-1} \left( A + \frac{N+2}{6} T_1(0) \right) \left( 1 - \frac{N+2}{48\pi} \frac{1}{\tilde{t}^{1/2}} \right) \\ &+ \frac{N+2}{6} K_1 \left( A + \frac{N+2}{6} T_1(0) \right) \\ &+ \ln g \left( B + \frac{N+2}{288\pi^2} \right) - \frac{N+2}{18} K_2 \\ &+ \tilde{t} - \frac{N+2}{24\pi} \sqrt{\tilde{t}} + \frac{N+2}{576\pi^2} \ln \tilde{t} + C + \frac{(N+2)^2}{1152\pi^2}. \end{aligned} \quad (18)$$

Cancellation of the terms of order  $1/g$  and  $\ln g$  gives

$$A = -\frac{N+2}{6} T_1(0) \quad B = -\frac{N+2}{288\pi^2}. \quad (19)$$

Finally, we compare (18) with the expression given in [18]:

$$\tilde{\chi} = \frac{1}{\tilde{t}} + \frac{N+2}{24\pi} \tilde{t}^{-3/2} - \frac{N+2}{576\pi^2} \frac{\ln \tilde{t}}{\tilde{t}^2} + \frac{E}{\tilde{t}^2}, \quad (20)$$

where  $E$  was computed by using resummed perturbation theory (in this case seven-loop expansions are available [13]). Comparing the two expressions we obtain

$$C = -E + \frac{N+2}{18} K_2 + \frac{(N+2)^2}{1152\pi^2}. \quad (21)$$

This expression shows that  $C$  is regularization dependent. To determine its explicit value, we compute the constant  $E$  using the results of [18], obtaining

$$E = -0.002504(6) \quad N = 1 \quad (22)$$

$$E = -0.002885(5) \quad N = 2 \quad (23)$$

$$E = -0.003042(3) \quad N = 3, \quad (24)$$

while for  $N \rightarrow \infty$ , we have  $E \approx N^2/(1152\pi^2) + O(N)$ . Correspondingly, if  $\hat{C} = C - \frac{N+2}{18} K_2$ , we have

$$\begin{aligned} \hat{C} &= 0.003296(6) \quad N = 1 \\ \hat{C} &= 0.004292(5) \quad N = 2 \\ \hat{C} &= 0.005241(3) \quad N = 3. \end{aligned} \quad (25)$$

For  $N \rightarrow \infty$ , the terms of order  $N^2$  cancel, so that  $\hat{C}$  is of order  $N$ .

It is interesting to extend the calculation to the continuum model in dimensional regularization, to compare with the result of [2] for  $N = 2$ . In this scheme  $T_1(0) = 0$ . Regularizing  $T_2$  in  $d = 3 - \epsilon$ , and renormalizing it by minimal subtraction, we obtain (we use the results of [21, 22])

$$K_2 = \frac{1}{16\pi^2} \ln \bar{\mu} + K_{20}, \quad (26)$$

where  $\bar{\mu}$  is the renormalization scale in the  $\overline{\text{MS}}$  scheme ( $\bar{\mu} = \sqrt{4\pi\mu} e^{-\gamma_E/2}$ ) and  $K_{20} = (1 - 2 \ln 3)/(32\pi^2) \approx -0.00379076$ . We can thus rewrite

$$\frac{\mu_{0c}^2}{g^2} = -\frac{N+2}{288\pi^2} \ln \frac{g}{\bar{\mu}} + \frac{N+2}{18} K_{20} + \hat{C} \quad (27)$$

We can compare this result with that reported in [2]. For  $N = 2$  we obtain  $\mu_{0c}^2/g^2 = 0.001904(5)$  for  $N = 2$  and  $g/\bar{\mu} = 3$ , to be compared with the numerical estimate 0.001920(2) of [2]. The two results are close, although they do not properly agree within errors (in any case, the difference is still acceptable being of the order of twice the sum of the error bars).

#### IV. CONCLUSIONS

We considered the  $O(N)$  invariant  $\phi^4$  theory in two and three dimensions and determined the nonperturbative mass renormalization  $\mu_{0c}^2$  one must perform to obtain the continuum limit of the model. Such a quantity is also relevant in the context of dilute relativistic and nonrelativistic Bose gases [1–4]. In two dimensions there are several computations [5–11] of  $\mu_{0c}^2$  for  $N = 1$ . In three dimensions it has been determined [2] for  $N = 2$ , the relevant case for Bose gases.

We computed the mass-renormalization constant  $\mu_{0c}^2$  for a generic lattice model in  $d = 2$  and  $d = 3$ . The necessary nonperturbative information was taken from Ref. [18], where several nonperturbative quantities were computed in the continuum limit (in that context the continuum limit was named critical crossover limit) as a function of the dimensionless renormalized mass. They were estimated by resumming the perturbative series in the massive renormalization scheme (four-loop [12] and seven-loop [12, 13] results are available in  $d = 2$  and  $d = 3$ , respectively), taking explicitly into account [14] the Borel summability of the perturbative series and the large-order behavior of their coefficients, determined by nonperturbative instanton calculations [15].

The results we obtain are in good agreement with present-day state-of-the-art numerical determinations, confirming the accuracy of resummed perturbation theory.

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